The University Of Sheffield.

DEPARTMENT OF PHYSICS & ASTRONOMY **Spring Semester 2010–11**

INTRODUCTION TO COSMOLOGY **2 hours**

Answer ALL questions in Section A and TWO questions in Section B.

Questions in Section A are marked out of 5, and those in Section B out of 15. The breakdown on the right-hand side of the page is meant as a guide to the marks that can be obtained from each part.

A formula sheet and list of physical constants is attached to this paper.

NOTE:

The Friedmann equation: $$\dot{a}(t)^2 = \frac{8\pi G}{3c^2}\left(\frac{\varepsilon_{r0}}{a(t)^2} - \frac{\varepsilon_{m0}}{a(t)}\right) - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3}a(t)^2$$

The fluid equation: $$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

The acceleration equation: $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

The Robertson-Walker metric: $$ds^2 = -c^2\,dt^2 + a(t)^2\left(dr^2 + x_k^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right)$$

$$\text{where } x_k(r) = \begin{cases} R\sin(r/R) & k = +1 \\ r & k = 0 \\ R\sinh(r/R) & k = -1 \end{cases}$$

SECTION A

Answer ALL questions in this section.

1. By considering the path of a photon, show that the comoving proper distance r is given by

$$r = c\int_{t_e}^{t_o} \frac{dt}{a(t)},$$

where t_e is the time at which the light was emitted and t_o is the time of observation. [$1\frac{1}{2}$]

Hence show that

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_o}{a(t_o)},$$

where λ_e, λ_o are the wavelengths at which the light is emitted and observed, respectively. [$3\frac{1}{2}$]

2. Define the density parameter Ω, and hence show that the Friedmann equation can be written in the form

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} + \frac{1-\Omega_0}{a^2} + \Omega_{\Lambda 0},$$

where the subscript 0 represents the present time and $\Omega_0 = \Omega_{r0} + \Omega_{m0} + \Omega_{\Lambda 0}$. [3]

Show that, in a universe with negligible radiation density, the expansion will be accelerating ($\ddot{a} > 0$) at any time at which $\Omega_\Lambda > \frac{1}{2}\Omega_m$. [2]

3. Using the boundary condition that $a = 1$ at $t = t_0$, show that a flat universe dominated by a cosmological constant can be described by

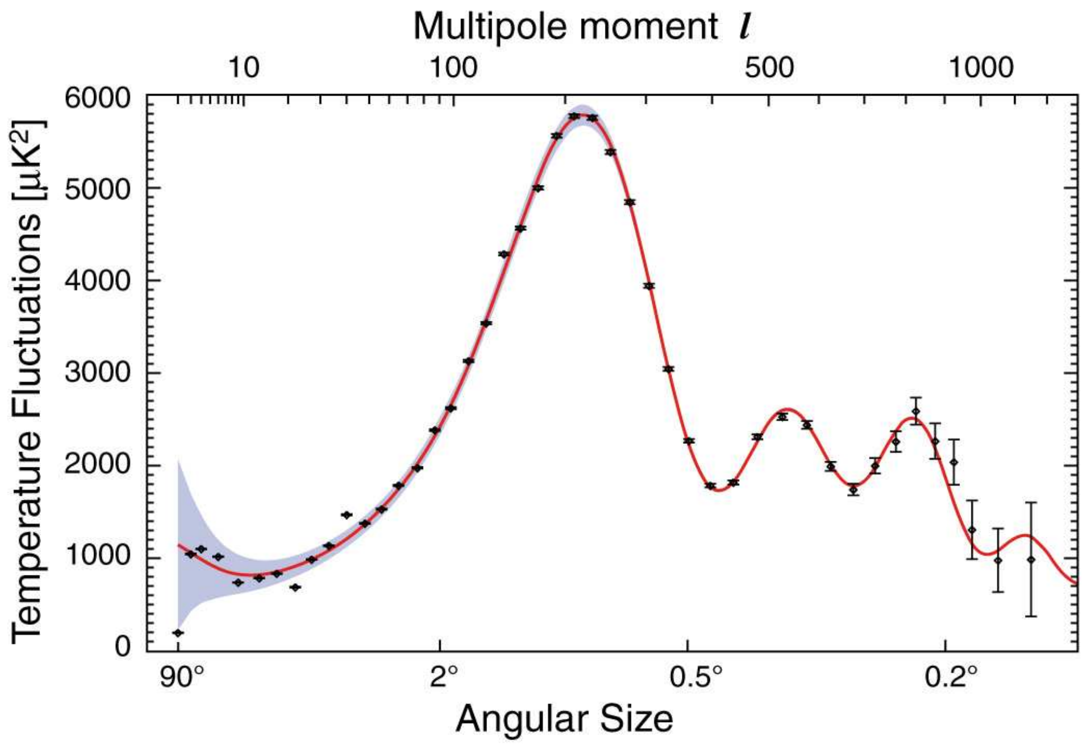
$$a(t) = \exp\left(H_0\left(t - t_0\right)\right),$$

where $H_0^2 = \Lambda/3$. [2]

Show that the comoving proper distance r in such a universe is given by cz/H_0. [2]

What is the horizon distance in this model? Briefly explain the physical significance of your answer. [1]

4. The figure below shows the power spectrum of the cosmic microwave background as measured by WMAP.



Explain how and why the appearance of this figure would change if

(a) the universe had significant positive curvature; $[1\frac{1}{2}]$

(b) the overall density were the same, but the ratio of baryonic to non-baryonic matter were higher; [2]

(c) the geometry were the same, but the overall matter density were higher. $[1\frac{1}{2}]$

(Note: **explain** the effect—do **not** simply say what will happen!)

SECTION B

Answer TWO questions in this section.

5. The current benchmark cosmological model has a flat geometry and negligible radiation density, but significant contributions from both $\Omega_{\rm m}$ and Ω_Λ. In this case, the Friedmann equation can be written

$$\dot{a}^2 = H_0^2\left(\frac{\Omega_{\rm m0}}{a} + (1-\Omega_{\rm m0})\,a^2\right).$$

The solution to this can be written in the parametric form

$$a(\theta) = \left(\frac{\Omega_{\rm m0}}{1-\Omega_{\rm m0}}\right)^{1/3}\tan^{2/3}\theta;$$

$$t(\theta) = \frac{2}{3H_0\sqrt{1-\Omega_{\rm m0}}}\ln\left|\sec\theta + \tan\theta\right|,$$

where θ is a dummy variable.

(a) Verify by differentiation that these parametric equations are indeed a solution of the above form of the Friedmann equation. [4]

[Note that the derivatives of $\sec\theta$ and $\tan\theta$ are $\sec\theta\tan\theta$ and $\sec^2\theta$ respectively.]

(b) Show that if $\Omega_{\rm m0} \to 1$, the parametric equations reduce to the form $a(t) = (t/t_0)^{2/3}$, where $H_0 t_0 = 2/3$. [4]

(c) According to WMAP, the best-fit parameters for this model are $\Omega_{\rm m0} = 0.266$, $H_0 = 71.0\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. For these values, determine:

(i) the redshift z at which the expansion switched from deceleration ($\ddot{a} < 0$) to acceleration; [2]

(ii) the age of the universe in Gyr; [2]

(iii) the look-back time for a quasar at redshift 3. [3]

6. For a universe which is not flat, it can be shown that

$$1-\Omega(t)=\begin{cases}(1-\Omega_0)\,a^2 & \text{radiation dominated}\\ (1-\Omega_0)\,a & \text{matter dominated}\\ (1-\Omega_0)\,/a^2 & \Lambda\text{ dominated}\end{cases}$$

where $\Omega=\Omega_\mathrm{r}+\Omega_\mathrm{m}+\Omega_\Lambda$ and the subscript 0 refers to the present time.

(a) If the universe is not assumed to be flat, WMAP's best fit gives $\Omega_0=1.08^{+0.09}_{-0.07}$ and $H_0=53\pm14\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. The redshift of the epoch of matter-radiation equality, z_eq, is found to be 3210±130.

(i) Neglecting the uncertainties in the fitted values of H_0 and z_eq, and assuming that the universe is matter-dominated from z_eq to the present time, calculate the bounds on $\Omega(z_\mathrm{eq})$ implied by the fitted value of Ω_0. [3]

(ii) For a matter-dominated universe, show that the angle subtended on the present-day sky by the horizon distance at time t is approximately $\sqrt{a}$. [3]

(iii) In this fit, the CMB is emitted (radiation decouples from matter) at $z=1091$ (the uncertainty here is negligible). Using the result from part (ii), determine the angle over which you might expect the CMB to be uniform in this model. [1]

[You may assume without proof that $a=(t/t_0)^{2/3}$ for a matter-only universe.]

(b) Using your answers to the previous parts, explain carefully why it is generally assumed that there must have been an epoch of inflation in the early universe. [3]

(c) Making the simplifying assumptions that the universe is currently 10^{10} years old and has been radiation-dominated throughout, calculate the expected value of $|1-\Omega(t)|$ at $t=10^{-35}$ s, assuming that at present $\Omega_0=1.08^{+0.09}_{-0.07}$ as found by WMAP. [2]

Hence calculate the value of Λ required to achieve this during inflation, if inflation starts at 10^{-35} s and ends at 10^{-33} s. Assume that $|1-\Omega|\sim1$ before the inflationary epoch. [3]

[You may assume without proof that $a=(t/t_0)^{1/2}$ for a radiation-only universe.]

7. (a) Define the terms **luminosity distance**, d_{L}, and **angular size distance**, d_{A}. [2]

(b) Show that $d_{\mathrm{L}} = (1+z)^2 d_{\mathrm{A}}$, for any redshift z and any value of curvature k. [4]

(c) Solve the Friedmann equation for a flat, matter-dominated universe, and hence show that, for such a universe, the proper distance d_{P} is given by

$$d_{\mathrm{P}} = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right),$$

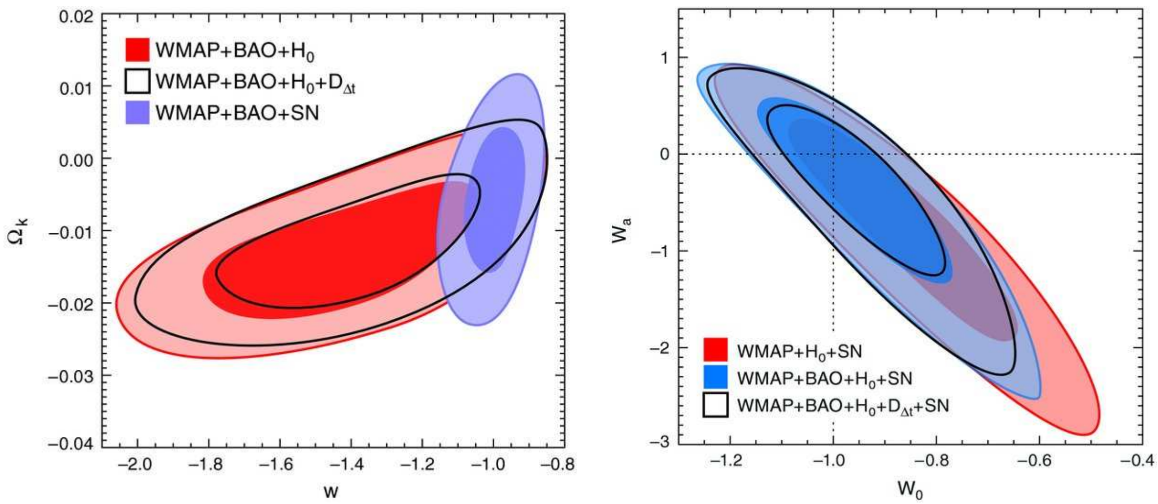
where z is the redshift of the object in question and H_0 is the present value of the Hubble parameter. [5]

(d) In a flat, matter-dominated universe with $H_0 = 50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, a radio galaxy is observed at redshift $z = 2.2$.

(i) Calculate the luminosity distance of the galaxy. If a Type Ia supernova of absolute magnitude –20 were to be observed in this galaxy, what would its apparent magnitude be? [2]

(ii) The galaxy is a double-lobed radio source similar to the local galaxy Cygnus A. Its radio lobes are separated by 150 kpc. What is the observed angular separation of the lobes in arc seconds? [2]

8. The figures below show results from the analysis of the 7-year WMAP data.



(a) All the analyses presented here include not only the WMAP data, but also data from other sources (galaxy redshift surveys for 'BAO' or 'baryon acoustic oscillations'; independent measurements of H_0; a time-delay distance to a gravitational lens, $D_{\Delta t}$; and results from Type Ia supernovae, SN). Explain why it is necessary to make use of other data samples in this way, even though the WMAP data are extremely precise (as can be seen from the figure in question 4). In your answer, mention why the specific data included in the figure have been chosen. [4]

(b) In the left-hand figure, the parameter w describes the equation of state of dark energy, $P = w\varepsilon$. Explain the cosmological significance of this parameter, why it is negative, and in particular the significance of the value $w = -1$ which seems to be preferred when the supernova data are included. [3]

(c) The inclusion of the supernova data seems to improve the sensitivity to w a great deal. Discuss reasons why this might be so. [4]

(d) The right-hand plot shows the results of a fit in which the dark-energy equation of state is assumed to vary with time, $w(a) = w_0 + w_a(1 - a)$, where $a(t)$ is the expansion parameter. Why do many theoretical cosmologists feel that an evolving equation of state for dark energy is desirable (given that the data are clearly consistent with $w_a = 0$)? [4]

END OF QUESTION PAPER