The University Of Sheffield.

Data Provided:
A formula sheet and table of physical constants is attached to this paper.

DEPARTMENT OF PHYSICS AND ASTRONOMY

Autumn Semester (2012-2013)

STELLAR STRUCTURE AND EVOLUTION **2 HOURS**

Answer THREE questions.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

1. A spherical star is in hydrostatic equilibrium. Assume that it is composed of an ideal gas of uniform mean particle mass, that there is negligible radiation pressure, and that the gas pressure vanishes at its surface. Prove the following results:

a) $$P_c > \frac{GM_s^2}{8\pi r_s^4},$$

b) $$\overline{T} > \frac{GM_s m}{6kr_s}.$$

P_c is the pressure at the centre of the star, M_s is the total mass of the star and r_s is the radius of the star. $\overline{T}$ is the mean temperature of the star. G, k and m are the gravitational constant, Boltzmann's constant and the mean particle mass, respectively. **[8]**

Use relation (b) to determine the minimum mean temperature of the Sun and then show that the assumption of negligible radiation pressure is valid at a typical point in the Sun. **[2]**

2. (a) A spherical star of homogeneous chemical composition is in a steady state. All energy transport is by radiation and radiation pressure is negligible. Write down the equations governing the structure of the star, defining all of the symbols carefully. **[2]**

(b) Consider a set of stars of various masses M_s which have the same chemical composition. Show that they obey a mass-luminosity relation $L_s \propto M_s^3$. Assume that the pressure is that of an ideal gas, that the opacity per unit mass, κ_0, is independent of density, ρ, and temperature, T, and that the rate of nuclear energy release per unit mass is $\varepsilon_0 \rho T^{17}$, where ε_0 is independent of ρ and T. **[8]**

3. (a) Derive the condition for the onset of convection in a stellar interior,

$$\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}P} > \frac{\gamma-1}{\gamma},$$

where P and T are the pressure and temperature at some radius r, and γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. **[6]**

(b) Explain briefly why, in a convective core, it is usually adequate to replace the above inequality with the equation

$$\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}P} = \frac{\gamma-1}{\gamma}.$$

[1]

(c) If radiation pressure is dominant instead of gas pressure, the above equation has γ replaced by $\frac{4}{3}$. If the opacity in such a case is a constant (κ_0), show that the energy carried by radiation (L_{rad}) in the core is given by

$$L_{\mathrm{rad}} = \frac{4\pi c\,GM}{\kappa_0},$$

where c is the velocity of light, G the gravitational constant and M the mass of material within r. **[3]**

4. With the aid of Hertzsprung-Russell (H-R) diagrams, describe the evolution of both a low-mass and a high-mass star from the time they join the main sequence to the cessation of nuclear fusion. **[10]**

5. (a) Describe what is meant by the term *degenerate gas*. **[1]**

(b) In what types of stars is matter degenerate? **[1]**

(c) The pressure, P, of any gas is

$$P = \frac{1}{3}\int_0^\infty N\,v\,p\;\mathrm{d}p,$$

where N is the number of particles per unit volume with momenta in the range $\mathrm{d}p$ around p, and v is the velocity corresponding to momentum p. Show that in a completely degenerate electron gas, neglecting interactions,

$$\begin{aligned} N = & \quad \frac{8\pi p^2}{h^3} \quad p \leq p_0, \\ = & \quad 0 \qquad p > p_0, \end{aligned}$$

where p_0 is the highest occupied momentum state and h is Planck's constant. **[2.5]**

(d) Hence find an expression for p_0 in terms of n_{e}, the number of electrons per unit volume. **[1]**

(e) Given that

$$v = \left(\frac{p}{m_{\mathrm{e}}}\right)\left(1 + \frac{p^2}{m_{\mathrm{e}}^2 c^2}\right)^{-1/2},$$

show that in the non-relativistic limit,

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{h^2}{m_{\mathrm{e}}}\right) n_{\mathrm{e}}^{5/3},$$

where m_{e} is the electron mass and c is the velocity of light. **[4.5]**

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS & MATHEMATICAL FORMULAE

Physical Constants

electron charge	$e = 1.60\times10^{-19}$ C
electron mass	$m_e = 9.11\times10^{-31}$ kg $= 0.511$ MeV c^{-2}
proton mass	$m_p = 1.673\times10^{-27}$ kg $= 938.3$ MeV c^{-2}
neutron mass	$m_n = 1.675\times10^{-27}$ kg $= 939.6$ MeV c^{-2}
Planck's constant	$h = 6.63\times10^{-34}$ J s
Dirac's constant ($\hbar = h/2\pi$)	$\hbar = 1.05\times10^{-34}$ J s
Boltzmann's constant	$k_B = 1.38\times10^{-23}$ J K^{-1} $= 8.62\times10^{-5}$ eV K^{-1}
speed of light in free space	$c = 299\,792\,458$ m s^{-1} $\approx 3.00\times10^{8}$ m s^{-1}
permittivity of free space	$\varepsilon_0 = 8.85\times10^{-12}$ F m^{-1}
permeability of free space	$\mu_0 = 4\pi\times10^{-7}$ H m^{-1}
Avogadro's constant	$N_A = 6.02\times10^{23}$ mol^{-1}
gas constant	$R = 8.32$ J mol^{-1} K^{-1}
ideal gas volume (STP)	$V_0 = 22.4$ l mol^{-1}
gravitational constant	$G = 6.67\times10^{-11}$ N m^2 kg^{-2}
Rydberg constant	$R_\infty = 1.10\times10^{7}$ m^{-1}
Rydberg energy of hydrogen	$R_H = 13.6$ eV
Bohr radius	$a_0 = 0.529\times10^{-10}$ m
Bohr magneton	$\mu_B = 9.27\times10^{-24}$ J T^{-1}
fine structure constant	$\alpha \approx 1/137$
Wien displacement law constant	$b = 2.898\times10^{-3}$ m K
Stefan's constant	$\sigma = 5.67\times10^{-8}$ W m^{-2} K^{-4}
radiation density constant	$a = 7.55\times10^{-16}$ J m^{-3} K^{-4}
mass of the Sun	$M_\odot = 1.99\times10^{30}$ kg
radius of the Sun	$R_\odot = 6.96\times10^{8}$ m
luminosity of the Sun	$L_\odot = 3.85\times10^{26}$ W
mass of the Earth	$M_\oplus = 6.0\times10^{24}$ kg
radius of the Earth	$R_\oplus = 6.4\times10^{6}$ m

Conversion Factors

1 u (atomic mass unit) $= 1.66\times10^{-27}$ kg $= 931.5$ MeV c^{-2}	1 Å (angstrom) $= 10^{-10}$ m
1 astronomical unit $= 1.50\times10^{11}$ m	1 g (gravity) $= 9.81$ m s^{-2}
1 eV $= 1.60\times10^{-19}$ J	1 parsec $= 3.08\times10^{16}$ m
1 atmosphere $= 1.01\times10^{5}$ Pa	1 year $= 3.16\times10^{7}$ s

Polar Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \mathrm{d}A = r\;\mathrm{d}r\,\mathrm{d}\theta$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Spherical Coordinates

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \qquad \mathrm{d}V = r^2\sin\theta\;\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Calculus

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$\tan x$	$\sec^2 x$
e^x	e^x	$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\ln x = \log_e x$	$\frac{1}{x}$	$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin x$	$\cos x$	$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$
$\cos x$	$-\sin x$	$\sinh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2+a^2}}$
$\cosh x$	$\sinh x$	$\cosh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2-a^2}}$
$\sinh x$	$\cosh x$	$\tanh^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2-x^2}$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x\;\cot x$	uv	$u'v + uv'$
$\sec x$	$\sec x\tan x$	u/v	$\frac{u'v-uv'}{v^2}$

Definite Integrals

$$\int_0^\infty x^n e^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}} \qquad (n \geq 0 \text{ and } a > 0)$$

$$\int_{-\infty}^{+\infty} e^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

Integration by Parts: $$\int_a^b u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\,\mathrm{d}x$$

Series Expansions

Taylor series: $f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$

Binomial expansion: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ and $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots \qquad (|x| < 1)$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\ln(1+x) = \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (|x| < 1)$

Geometric series: $\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$

Stirling's formula: $\log_e N! = N \log_e N - N$ or $\ln N! = N \ln N - N$

Trigonometry

$\sin(a \pm b) = \sin a \ \cos b \pm \cos a \ \sin b$

$\cos(a \pm b) = \cos a \ \cos b \mp \sin a \ \sin b$

$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \ \tan b}$

$\sin 2a = 2 \sin a \ \cos a$

$\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$

$\sin a + \sin b = 2 \sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b)$

$\sin a - \sin b = 2 \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b)$

$\cos a + \cos b = 2 \cos \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b)$

$\cos a - \cos b = -2 \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b)$

$e^{i\theta} = \cos\theta + i \ \sin\theta$

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

$\cosh\theta = \frac{1}{2}\left(e^{\theta} + e^{-\theta}\right)$ and $\sinh\theta = \frac{1}{2}\left(e^{\theta} - e^{-\theta}\right)$

Spherical geometry: $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$ and $\cos a = \cos b \ \cos c + \sin b \ \sin c \ \cos A$

Vector Calculus

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z = A_j B_j$$

$$\mathbf{A}\times\mathbf{B} = (A_y B_z - A_z B_y)\,\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\,\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\,\hat{\mathbf{k}} = \epsilon_{ijk} A_j B_k$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\text{grad}\,\phi = \nabla\phi = \partial_j\phi = \frac{\partial\phi}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{k}}$$

$$\text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \partial_j A_j = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \epsilon_{ijk}\partial_j A_k = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{k}}$$

$$\nabla\cdot\nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla\times(\nabla\phi) = 0 \qquad \text{and} \qquad \nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$