Data Provided:
A formula sheet and table of physical constants is attached to this paper.

DEPARTMENT OF PHYSICS AND ASTRONOMY

Autumn (2015)

SOLID STATE PHYSICS

The paper is divided into 5 questions.

Answer compulsory question 1, which is marked out of 20.

Answer any two out of the optional questions 2-5, each of which is marked out of 15.

The breakdown on the right hand-side of the paper is meant as a guide to the marks that can be obtained from each part.

Please clearly indicate the question numbers on which you would like to be examined on the front cover of your answer book. Cross through any work that you do not wish to be examined.
Question 1

COMPULSORY

a) The expression \(2\mathbf{k} \cdot \mathbf{G} = G^2\) describes the condition for Bragg diffraction in a periodic lattice, where \(\mathbf{k}\) is the electron wavevector and \(\mathbf{G}\) is a reciprocal lattice vector. Using this expression, deduce the forms of the first and second Brillouin zones for a two dimensional square lattice. [3]

b) Show that the Fermi energy of conduction electrons is given by \(E_F = \frac{\hbar^2}{2m} \left(3\pi^2 n\right)^{2/3}\), where \(n\) is the electron density and \(m\) is the electron mass. [3]

c) A metal has body-centred-cubic structure and lattice constant of 3.5 x 10^{-10} m. Each atom contributes one electron to conduction. Using the expression for the Fermi energy from part b), deduce the values of the Fermi energy, Fermi temperature and Fermi velocity. You may take \(m\) to be equal to the free electron mass. [3]

d) Sketch a graph showing variation of carrier concentration in a doped semiconductor as a function of temperature. Explain the temperature behaviour of the concentration at low, intermediate and high temperatures. [3]

e) Where is the Fermi level located for the case of a semiconductor doped with acceptors? Assume that the temperature is very low \(k_B T < E_A\) where \(E_A\) is the binding energy of the acceptor. [1]

f) Explain the physical basis of paramagnetism. In your explanation discuss the temperature dependence of magnetic susceptibility in paramagnetic materials. [3]

g) Co^{2+} ions have electronic configuration 3d^{7}. Use Hund’s rule to calculate the quantum numbers \(J, L\) and \(S\). Determine the maximum value of the magnetic moment of the ion along an arbitrary axis in space in the units of the Bohr magneton. In your answer you may use the expression for the Landé \(g\)-factor

\[
g = \left(\frac{3}{2} + \frac{S(S+1)-L(L+1)}{2J(J+1)}\right).\]  [2]

h) Explain the concept of plasmons in metals. Briefly outline an experiment which enables the observation of plasmons in metallic films. [2]
Question 2

OPTIONAL

a) Using a tight binding model, deduce that the number of states in a band is given by $2N$, where $N$ is the number of atoms. [2]

b) Explain why different bands in solids have different widths in energy. How do you expect the widths of the bands to vary as a function of energy, from low energy bands to high energy bands. [2]

c) Describe the physical basis of a technique which may be used to probe the electron occupancy of the conduction band in a metal, by probing transitions between different bands. [3]

d) Electrons in a metal have mobility $1 \text{ m}^2/\text{Vs}$, carrier density $3 \times 10^{28} \text{ m}^{-3}$, and mass equal to the free electron mass.

   (i) Calculate the conductivity of the metal. [2]

   (ii) An electric field of 10V/m is applied. Calculate the displacement in $k$-space of the Fermi distribution which results. [2]

   (iii) How does this displacement compare to the likely value of the Fermi wavevector in the metal. A derivation of the value for the Fermi wavevector is not required. [1]

e) Describe two types of scattering mechanism which lead to resistance to electron flow in metals. [2]

f) Explain why the two mechanisms in e) have very different temperature dependences. [1]
Question 3

OPTIONAL

a) Provide a qualitative explanation of why the binding energy of an electron to a donor, which is of the order of 10’s of meV in the most common semiconductors (Si, Ge or GaAs), is much smaller than the binding energy of an electron (13.6 eV) in a hydrogen atom. How does the Bohr radius of an electron to a donor in semiconductors compare to the Bohr radius of an electron in a hydrogen atom? [3]

b) In a heavily doped sample of GaAs the onset of optical absorption occurs at a wavelength of 700 nm at 300 K. Calculate the energy of the Fermi level above the minimum of the conduction band. The electron \((m_e)\) and hole \((m_h)\) effective masses in GaAs are 0.067 \(m_0\) and 0.45 \(m_0\), respectively, where \(m_0\) is the free electron mass. The band gap of GaAs is 1.43 eV at 300 K. [4]

c) Explain the concept of a Wannier-Mott exciton in a semiconductor material. Explain why the approximation of the effective mass and macroscopic dielectric function is valid in the calculation of Wannier-Mott exciton binding energies using a hydrogenic model. [2]

d) A piece of ferric oxide is placed in an external magnetic field \(H\). The resultant magnetic field \(B\) inside the material is 1.258 T and its magnetisation \(M\) is 1500 A/m. Calculate the magnetic susceptibility of ferric oxide and the value of the \(H\)-field inside the material. [3]

e) Explain how and why the phenomenon of impurity compensation may arise in a semiconductor initially doped only with donors or acceptors. [3]
Question 4

OPTIONAL

a) Derive the following expression for electron concentration \( n_e \) in the conduction band as a function of temperature \( T \) in an undoped semiconductor:

\[
n_e = 2 \left( \frac{2\pi m_e^* k_B T}{\hbar^2} \right)^{3/2} \exp\left(\frac{E_F - E_g}{k_B T}\right).
\]

You may use the following expressions for the electron density of states \( g_e \) per unit volume and the electron Fermi-Dirac distribution \( f(E) \):

\[
g_e = \frac{4\pi(2m_e^*)^{3/2}}{\hbar^3} \sqrt{E - E_g},
\]

\[
f(E) = \frac{1}{1 + \exp\left(\frac{E - E_F}{k_B T}\right)}.
\]

Here \( m_e^* \) is the electron effective mass, \( E_F \) is the Fermi level, \( E_g \) is the band gap and \( E \) is the electron energy, which is set to zero for electrons in the valence band with zero momentum. In your derivation you may assume that \(|E - E_F| >> k_B T\) and use the following integral \( \int_0^\infty x^{3/2} e^{-x} dx = \sqrt{\pi}/2 \). [6]

b) A sample of silicon doped with donors ceases to show intrinsic behaviour at temperatures below 360 K. Estimate the donor concentration. The intrinsic carrier concentration in silicon at 300 K is \( 2 \times 10^{16} \text{ m}^{-3} \). The value of the band gap in Si is 1.1 eV. [3.5]

c) Briefly explain the three mechanisms which contribute to the total energy of a ferromagnetic crystal, where a stable spontaneous domain configuration is developed. How do these mechanisms compete with each other? [3]

d) Calculate the average magnetic moment along the field direction per atom at 300 K when a paramagnetic material having only spin (i.e. \( L = 0 \)) is subjected to a uniform \( H \)-field of \( 10^6 \text{ A/m} \). You may use the following formula for magnetic susceptibility of a paramagnet

\[
\chi = \frac{\mu_0 N g^2 J(J+1)\mu_B^2}{3kT},
\]

where \( g \) is the Lande factor, \( J \) is the total angular momentum quantum number, \( \mu_B \) is the Bohr magneton. Express your answer in units of the Bohr magneton. [2.5]
OPTIMAL

Question 5

a) Sketch a schematic diagram of the experiment enabling the observation of the Hall effect. Starting from the equation of motion for electrons in crossed magnetic ($B$) and electric ($E$) fields show that the Hall electric field in a metal is given by $|E_H| = \frac{e\tau m_e}{e} E$, where $\tau$, $m_e$, $e$ are the electron scattering time, effective mass and charge, respectively. [5]

b) Explain how the sign of the main charge carriers in a semiconductor material can be obtained from cyclotron resonance and Hall experiments. [2]

c) (i) An electron cyclotron resonance is observed at frequency $f = 100$ GHz for a semiconductor sample placed in an external magnetic field. Find the value of the magnetic field. The effective mass of the electron in the sample is $m_e = 0.07 m_0$, where $m_0$ is the free electron mass. [2.5]

(ii) Estimate the minimum electron scattering time required to observe cyclotron resonance at 200 GHz. [2.5]

d) Ferromagnetic nickel has a Curie Temperature of 510 K and a magnetic moment of 0.60$\mu_B$ per ion ($\mu_B$ is the Bohr magneton). Calculate the value of the effective “internal” magnetic field that is responsible for the spontaneous magnetisation of nickel. Comment on the main physical mechanism responsible for this field. [3]

END OF EXAMINATION PAPER
### Physical Constants

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>electron charge</td>
<td>$e = 1.60 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>electron mass</td>
<td>$m_e = 9.11 \times 10^{-31}$ kg = 0.511 MeV $c^{-2}$</td>
</tr>
<tr>
<td>proton mass</td>
<td>$m_p = 1.673 \times 10^{-27}$ kg = 938.3 MeV $c^{-2}$</td>
</tr>
<tr>
<td>neutron mass</td>
<td>$m_n = 1.675 \times 10^{-27}$ kg = 939.6 MeV $c^{-2}$</td>
</tr>
<tr>
<td>Planck’s constant</td>
<td>$h = 6.63 \times 10^{-34}$ J s</td>
</tr>
<tr>
<td>Dirac’s constant ($h = h/2\pi$)</td>
<td>$\hbar = 1.05 \times 10^{-34}$ J s</td>
</tr>
<tr>
<td>Boltzmann’s constant</td>
<td>$k_B = 1.38 \times 10^{-23}$ J K$^{-1}$ = 8.62 $\times 10^{-5}$ eV K$^{-1}$</td>
</tr>
<tr>
<td>speed of light in free space</td>
<td>$c = 299 792 458$ m s$^{-1}$ ≈ 3.00 $\times 10^8$ m s$^{-1}$</td>
</tr>
<tr>
<td>permittivity of free space</td>
<td>$\varepsilon_0 = 8.85 \times 10^{-12}$ F m$^{-1}$</td>
</tr>
<tr>
<td>permeability of free space</td>
<td>$\mu_0 = 4\pi \times 10^{-7}$ H m$^{-1}$</td>
</tr>
<tr>
<td>Avogadro’s constant</td>
<td>$N_A = 6.02 \times 10^{23}$ mol$^{-1}$</td>
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<tr>
<td>gas constant</td>
<td>$R = 8.314$ J mol$^{-1}$ K$^{-1}$</td>
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<tr>
<td>ideal gas volume (STP)</td>
<td>$V_0 = 22.4$ L mol$^{-1}$</td>
</tr>
<tr>
<td>gravitational constant</td>
<td>$G = 6.67 \times 10^{-11}$ N m$^2$ kg$^{-2}$</td>
</tr>
<tr>
<td>Rydberg constant</td>
<td>$R_\infty = 1.10 \times 10^{7}$ m$^{-1}$</td>
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<tr>
<td>Rydberg energy of hydrogen</td>
<td>$R_H = 13.6$ eV</td>
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<tr>
<td>Bohr radius</td>
<td>$a_0 = 0.529 \times 10^{-10}$ m</td>
</tr>
<tr>
<td>Bohr magneton</td>
<td>$\mu_B = 9.27 \times 10^{-24}$ J T$^{-1}$</td>
</tr>
<tr>
<td>fine structure constant</td>
<td>$\alpha \approx 1/137$</td>
</tr>
<tr>
<td>Wien displacement law constant</td>
<td>$b = 2.898 \times 10^{-3}$ m K</td>
</tr>
<tr>
<td>Stefan’s constant</td>
<td>$\sigma = 5.67 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$</td>
</tr>
<tr>
<td>radiation density constant</td>
<td>$a = 7.55 \times 10^{-16}$ J m$^{-3}$ K$^{-4}$</td>
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<tr>
<td>mass of the Sun</td>
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<tr>
<td>radius of the Sun</td>
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<tr>
<td>luminosity of the Sun</td>
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</tr>
<tr>
<td>mass of the Earth</td>
<td>$M_\oplus = 6.0 \times 10^{24}$ kg</td>
</tr>
<tr>
<td>radius of the Earth</td>
<td>$R_\oplus = 6.4 \times 10^6$ m</td>
</tr>
</tbody>
</table>

### Conversion Factors

- 1 u (atomic mass unit) = $1.66 \times 10^{-27}$ kg = 931.5 MeV $c^{-2}$
- 1 Å (angstrom) = $10^{-10}$ m
- 1 astronomical unit = $1.50 \times 10^{11}$ m
- 1 eV = $1.60 \times 10^{-19}$ J
- 1 atmosphere = $1.01 \times 10^5$ Pa
- 1 g (gravity) = 9.81 m s$^{-2}$
- 1 parsec = $3.08 \times 10^{16}$ m
- 1 year = $3.16 \times 10^7$ s
Polar Coordinates

\[ x = r \cos \theta \quad y = r \sin \theta \quad dA = r \, dr \, d\theta \]

\[ \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \]

Spherical Coordinates

\[ x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta \quad dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \]

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

Calculus

| \( f(x) \) | \( f'(x) \) | \( f(x) \) | \( f'(x) \) |
| \( x^n \) | \( n x^{n-1} \) | \( \tan x \) | \( \sec^2 x \) |
| \( e^x \) | \( e^x \) | \( \sin^{-1} \left( \frac{x}{a} \right) \) | \( \frac{1}{\sqrt{a^2 - x^2}} \) |
| \( \ln x = \log_e x \) | \( \frac{1}{x} \) | \( \cos^{-1} \left( \frac{x}{a} \right) \) | \( -\frac{1}{\sqrt{a^2 - x^2}} \) |
| \( \sin x \) | \( \cos x \) | \( \tan^{-1} \left( \frac{x}{a} \right) \) | \( \frac{a}{a^2 + x^2} \) |
| \( \cos x \) | \( -\sin x \) | \( \sinh^{-1} \left( \frac{x}{a} \right) \) | \( \frac{1}{\sqrt{x^2 + a^2}} \) |
| \( \cosh x \) | \( \sinh x \) | \( \cosh^{-1} \left( \frac{x}{a} \right) \) | \( \frac{1}{\sqrt{x^2 - a^2}} \) |
| \( \sinh x \) | \( \cosh x \) | \( \tanh^{-1} \left( \frac{x}{a} \right) \) | \( \frac{a}{a^2 - x^2} \) |
| \( \cosec x \) | \( -\cosec x \cot x \) | \( u/v \) | \( u'v + uv' \) |
| \( \sec x \) | \( \sec x \tan x \) | \( u/v \) | \( u/v \) |

Definite Integrals

\[ \int_0^\infty x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}} \quad (n \geq 0 \text{ and } a > 0) \]

\[ \int_{-\infty}^{+\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}} \]

\[ \int_{-\infty}^{+\infty} x^2 e^{-ax^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}} \]

Integration by Parts:

\[ \int_a^b u(x) \frac{dv(x)}{dx} \, dx = u(x)v(x) \bigg|_a^b - \int_a^b \frac{du(x)}{dx} v(x) \, dx \]
### Series Expansions

**Taylor series:** 
\[ f(x) = f(a) + \frac{(x - a)}{1!} f'(a) + \frac{(x - a)^2}{2!} f''(a) + \frac{(x - a)^3}{3!} f'''(a) + \cdots \]

**Binomial expansion:** 
\[ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \text{and} \quad \binom{n}{k} = \frac{n!}{(n-k)!k!} \]
\[(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots \quad (|x| < 1)\]
\[e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \]
\[\ln(1 + x) = \log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad (|x| < 1)\]

**Geometric series:** 
\[\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}\]

**Stirling’s formula:** 
\[\log_e N! = N \log_e N - N \quad \text{or} \quad \ln N! = N \ln N - N\]

### Trigonometry

\[\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b\]
\[\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b\]
\[\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}\]
\[\sin 2a = 2 \sin a \cos a\]
\[\cos 2a = \cos^2 a - \sin^2 a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a\]
\[\sin a + \sin b = 2 \sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)\]
\[\sin a - \sin b = 2 \cos \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)\]
\[\cos a + \cos b = 2 \cos \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)\]
\[\cos a - \cos b = -2 \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)\]
\[e^{i\theta} = \cos \theta + i \sin \theta\]
\[\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \quad \text{and} \quad \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})\]
\[\cosh \theta = \frac{1}{2} (e^{\theta} + e^{-\theta}) \quad \text{and} \quad \sinh \theta = \frac{1}{2} (e^{\theta} - e^{-\theta})\]

**Spherical geometry:** 
\[\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \quad \text{and} \quad \cos a = \cos b \cos c + \sin b \sin c \cos A\]
Vector Calculus

\[ \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = A_j B_j \]

\[ \mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k} = \epsilon_{ijk} A_j B_k \]

\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C} \]

\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = B \cdot (\mathbf{C} \times \mathbf{A}) = C \cdot (\mathbf{A} \times \mathbf{B}) \]

\[ \text{grad} \phi = \nabla \phi = \partial_j \phi \hat{i} + \partial \phi \hat{j} + \partial \phi \hat{k} \]

\[ \text{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \partial_j A_j = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]

\[ \text{curl} \mathbf{A} = \nabla \times \mathbf{A} = \epsilon_{ijk} \partial_j A_k = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{j} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{k} \]

\[ \nabla \cdot \nabla \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]

\[ \nabla \times (\nabla \phi) = 0 \quad \text{and} \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \]

\[ \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]