PHY202 – Quantum Mechanics
Homework 2

1. A particle of mass m is prepared in the time dependent state

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_0(x)\exp\left(-iE_0t/\hbar\right) + \psi_1(x)\exp\left(-iE_1t/\hbar\right)\right],$$

where $\psi_0(x)$ and $\psi_1(x)$ are the ground and the first excited normalised eigenstate of the linear harmonic oscillator, $\psi_0(x) = A_0\exp\left(-\frac{x^2}{2a^2}\right)$ and $\psi_1(x) = A_1 x\exp\left(-\frac{x^2}{2a^2}\right)$, where $A_0 = \left(\frac{1}{\pi a^2}\right)^{1/4}$, $A_1 = \left(\frac{4}{\pi a^6}\right)^{1/4}$ and $a^2 = \hbar/m\omega$. $E_0 = \frac{1}{2}\hbar\omega$ and $E_1 = \frac{3}{2}\hbar\omega$ are the corresponding energy eigenvalues.

(a) Show that the expectation value of momentum for $\Psi(x,t)$ is given by [3]

$$\langle p(t)\rangle = -\sqrt{\frac{m\hbar\omega}{2}}\sin\omega t.$$

(b) Compute the expectation value of the total energy $\langle E(t)\rangle$ and of the potential energy $\langle V(t)\rangle$ for $\Psi(x,t)$. Hence compute the expectation value of the kinetic energy $\langle T(t)\rangle$ and show that it is given by $\langle T(t)\rangle = \hbar\omega/2$. Compare it to $\langle E(t)\rangle$ and $\langle V(t)\rangle$. [2]

(c) Using the relation $\langle p^2(t)\rangle = 2m\langle T(t)\rangle$ or otherwise, show that the uncertainty in momentum is given by [1]

$$\Delta p(t) = \sqrt{m\hbar\omega}\sqrt{1 - \frac{1}{2}\sin^2\omega t}.$$

You can use the integrals $\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2/a^2} = a\sqrt{\pi}$, $\int_{-\infty}^{\infty} \mathrm{d}x\, x^2\exp\left(-x^2/a^2\right) = \frac{1}{2}\sqrt{\pi}a^3$ and $\int_{-\infty}^{\infty} \mathrm{d}x\, x^4\exp\left(-x^2/a^2\right) = \frac{3}{4}\sqrt{\pi}a^5$. Note $\int_{-\infty}^{\infty} \mathrm{d}x\, x^n\exp\left(-x^2/a^2\right) = 0$ when n is an odd integer. (A suggestion: express $\Psi(x,t)$ as
$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_0(x)\epsilon_0(t) + \psi_1(x)\epsilon_1(t)\right]$, where $\epsilon_n(t) = \exp\left(-iE_nt/\hbar\right)$ and $n = 0, 1$.)

2. A beam of particles of mass m and energy $E > 0$ moving in the negative x direction is incident at $x = 0$ on a step given by

$$V(x) = \begin{cases} 0 & x < 0 \quad \text{region 1} \\ -V_0 & x \geq 0 \quad \text{region 2,} \end{cases}$$

where $V_0 > 0$.

(a) For $x > 0$ (region 2) write down the time independent Schrödinger equation and find its solutions and the associated wavenumbers. What condition must the total energy E satisfy for the beam to propagate? [1]

(b) Find the analogous solution for $x < 0$ (region 1). [1]

(c) State the boundary conditions that the wavefunctions must satisfy at $x = 0$ and compute the total currents on the right side and on the left side of step. Hence compute the probability of reflection P_R and the probability of transmission P_T. [2]

Due Wednesday 8 December 2010, 17.00.

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