Topic 9: Quantum propagation

Outline:

- wavepacket propagation
- quantum mechanical currents
- potential step
- potential barrier
- quantum mechanical tunnelling
- the HUP for time and energy
- QM tunnelling in nature

Wavepacket propagation

Wavepacket propagation

Recall

- a wavepacket consisting of a continuous spectrum of momenta p

e.g., unbound (free) states, or bound states in x-dependent potential $V(x)$

$$\boxed{\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) \exp\left(\frac{i}{\hbar} px\right)}$$

$\phi(p)$ – (in general complex) momentum profile

Wavepacket propagation

Recall

- a wavepacket consisting of a continuous spectrum of momenta p

e.g., unbound (free) states, or bound states in x-dependent potential $V(x)$

$$\boxed{\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) \exp\left(\frac{i}{\hbar} px\right)}$$

$\phi(p)$ – (in general complex) momentum profile

- The full time dependent wavefunction is given by

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) \exp\left(\frac{i}{\hbar} px\right) \exp\left(-\frac{i}{\hbar} Et\right)}$$

where $E = p^2/2m + V(x)$ is the total energy

Wavepacket propagation

Recall

- a wavepacket consisting of a continuous spectrum of momenta p

  e.g., unbound (free) states, or bound states in x-dependent potential $V(x)$

$$\boxed{\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) \exp\left(\frac{i}{\hbar} p x\right)}$$

  $\phi(p)$ – (in general complex) momentum profile

- The full time dependent wavefunction is given by

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) \exp\left(\frac{i}{\hbar} p x\right) \exp\left(-\frac{i}{\hbar} E t\right)}$$

  where $E = p^2/2m + V(x)$ is the total energy

- Time evolution of $\Psi(x,t)$ is in general very complicated, even for fairly simple cases like Gaussian profiles.
  Below we will consider some simplified cases.

Wavepacket propagation

Recall

- a wavepacket consisting of a continuous spectrum of momenta p

e.g., unbound (free) states, or bound states in x-dependent potential $V(x)$

$$\boxed{\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) \exp\left(\frac{i}{\hbar} px\right)}$$

$\phi(p)$ – (in general complex) momentum profile

- The full time dependent wavefunction is given by

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) \exp\left(\frac{i}{\hbar} px\right) \exp\left(-\frac{i}{\hbar} Et\right)}$$

where $E = p^2/2m + V(x)$ is the total energy

- Time evolution of $\Psi(x,t)$ is in general very complicated, even for fairly simple cases like Gaussian profiles.
  Below we will consider some simplified cases.

First let's introduce some useful and general language.

Quantum mechanical current

Quantum mechanical current

- Recall, the probability density is given by

$$P(x,t) = |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$$

Quantum mechanical current

- Recall, the probability density is given by

$$P(x,t) = |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$$

- introduce a probability current $j(x,t)$ (or probability flux)

$$\boxed{j(x,t) = -i\frac{\hbar}{2m}\left[\Psi^*(x,t)\frac{\partial\Psi(x,t)}{\partial\, x} - \Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial\, x}\right]}$$

Sign convention: $j(x,t) > 0$ (< 0) means flux moving right (left).

Quantum mechanical current

- Recall, the probability density is given by

$$P(x,t) = |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$$

- introduce a probability current $j(x,t)$ (or probability flux)

$$\boxed{j(x,t) = -i\frac{\hbar}{2m}\left[\Psi^*(x,t)\frac{\partial\Psi(x,t)}{\partial x} - \Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial x}\right]}$$

  Sign convention: $j(x,t) > 0$ (< 0) means flux moving right (left).

- if $\Psi(x,t)$ is a solution of the TDSE, then $P(x,t)$ and $j(x,t)$ satisfy a continuity equation

$$\boxed{\frac{\partial P(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}}$$

Quantum mechanical current

- Recall, the probability density is given by

$$P(x,t) = |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$$

- introduce a probability current $j(x,t)$ (or probability flux)

$$\boxed{j(x,t) = -i\frac{\hbar}{2m}\left[\Psi^*(x,t)\frac{\partial\Psi(x,t)}{\partial x} - \Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial x}\right]}$$

  Sign convention: $j(x,t) > 0$ (< 0) means flux moving right (left).

- if $\Psi(x,t)$ is a solution of the TDSE, then $P(x,t)$ and $j(x,t)$ satisfy a continuity equation

$$\boxed{\frac{\partial P(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}}$$

- Physical interpretation: conservation of probability.
  The change of the probability density at x is equal to its “outflow” from there.
- Analogous to: electric charge current, gas density and flow, etc.

Examples

Examples

- take a stationary state $\Psi(x,t) = \psi(x)\exp\left(-\frac{i}{\hbar}Et\right)$, with $\psi(x)$ satisfying the TISE and total energy E. Then

$$\boxed{j(x,t) = -i\frac{\hbar}{2m}\left[\psi^*(x)\frac{\partial\psi(x)}{\partial\, x} - \psi(x)\frac{\partial\psi^*(x)}{\partial\, x}\right]}$$

Note $j(x)$ is independent of time! Interpretation: constant flux.

Examples

- take a stationary state $\Psi(x,t) = \psi(x)\exp\left(-\frac{i}{\hbar}Et\right)$, with $\psi(x)$ satisfying the TISE and total energy E. Then

$$\boxed{j(x,t) = -i\frac{\hbar}{2m}\left[\psi^*(x)\frac{\partial\psi(x)}{\partial\,x} - \psi(x)\frac{\partial\psi^*(x)}{\partial\,x}\right]}$$

  Note $j(x)$ is independent of time! Interpretation: constant flux.

- take $\psi(x)$ to be a momentum eigenfunction $\psi_p(x) = \exp\left(\frac{i}{\hbar}px\right)$,

$$\boxed{j(x,t) = -i\frac{\hbar}{2m}\left[\psi^*(x)\frac{\partial\psi(x)}{\partial\,x} - \psi(x)\frac{\partial\psi^*(x)}{\partial\,x}\right] = \frac{p}{m}}$$

  Interpretation: classically $\frac{p}{m} = v$ is velocity.

Examples

- take a stationary state $\Psi(x,t) = \psi(x) \exp\left(-\frac{i}{\hbar} E t\right)$, with $\psi(x)$ satisfying the TISE and total energy E. Then

  $$\boxed{j(x,t) = -i\frac{\hbar}{2m}\left[\psi^*(x)\frac{\partial\psi(x)}{\partial\, x} - \psi(x)\frac{\partial\psi^*(x)}{\partial\, x}\right]}$$

  Note $j(x)$ is independent of time! Interpretation: constant flux.

- take $\psi(x)$ to be a momentum eigenfunction $\psi_p(x) = \exp\left(\frac{i}{\hbar} p x\right)$,

  $$\boxed{j(x,t) = -i\frac{\hbar}{2m}\left[\psi^*(x)\frac{\partial\psi(x)}{\partial\, x} - \psi(x)\frac{\partial\psi^*(x)}{\partial\, x}\right] = \frac{p}{m}}$$

  Interpretation: classically $\frac{p}{m} = v$ is velocity.

- take $\psi(x)$ to be a real function (like in bound states)

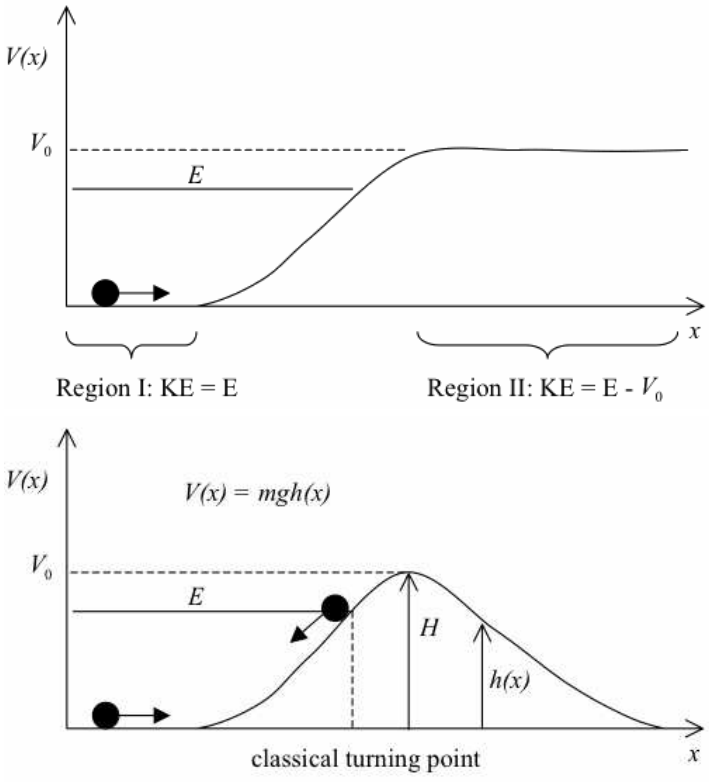
  $$\boxed{j(x,t) = 0}$$

  Interpretation: a bound state of definite energy is a standing wave.

Classical Potential Steps and Barriers

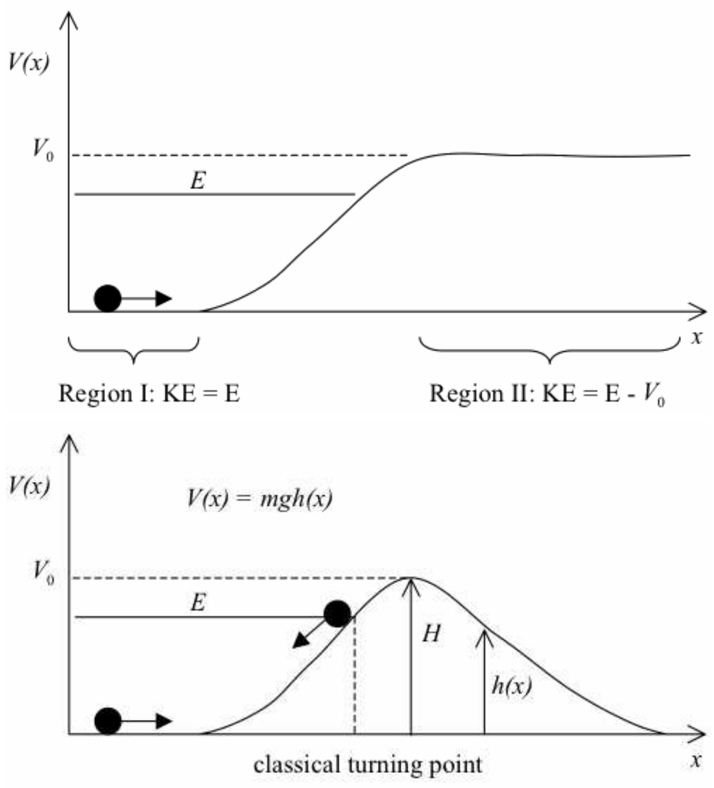
Classical Potential Steps and Barriers

Consider a flux of particles incident from the left on the potential step or barrier.



Classical Potential Steps and Barriers

Consider a flux of particles incident from the left on the potential step or barrier.

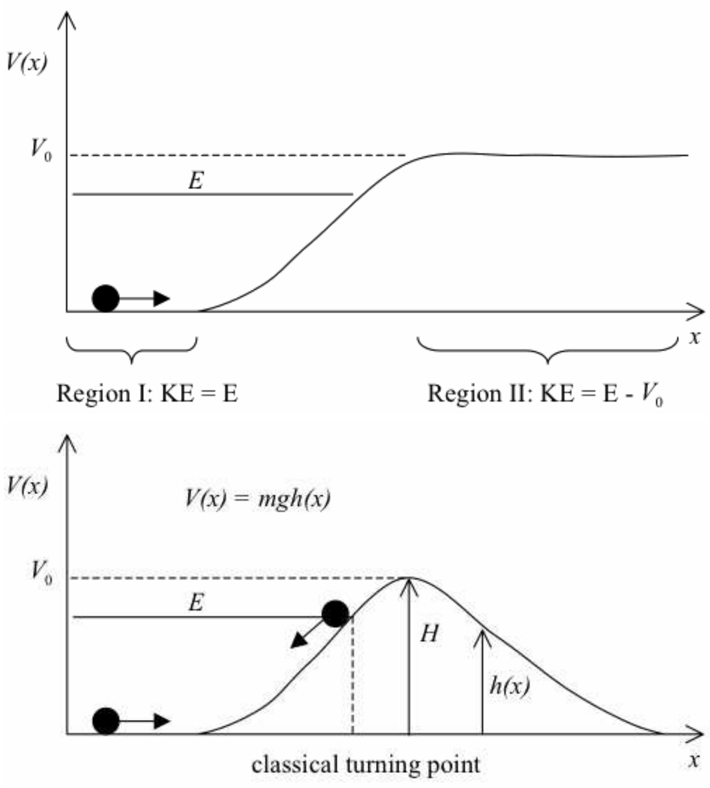


Classically:

- if $E > V_0$, all the particles will pass over the step/barrier (they are transmitted)
- they will slow down (smaller momentum)
- if $E < V_0$, all the particles are reflected

Classical Potential Steps and Barriers

Consider a flux of particles incident from the left on the potential step or barrier.



Classically:

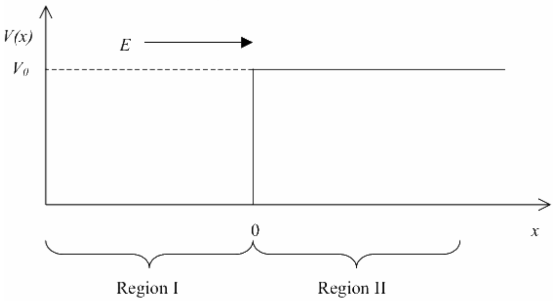
- if $E > V_0$, all the particles will pass over the step/barrier (they are transmitted)
- they will slow down (smaller momentum)
- if $E < V_0$, all the particles are reflected

In both cases the flux of particles must be the same. (They don't sink anywhere.)

Quantum mechanical potential step

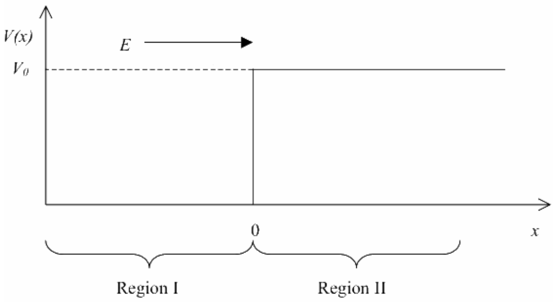
Quantum mechanical potential step

Consider a flux of particles with total energy E incident from the left on a square potential step. Assume $E > V_0$.



Quantum mechanical potential step

Consider a flux of particles with total energy E incident from the left on a square potential step. Assume $E > V_0$.



- $x < 0$ (Region I):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

$$\Rightarrow \psi''(x) = -k_1^2\psi(x) \quad k_1^2 = \frac{2mE}{\hbar^2} > 0$$

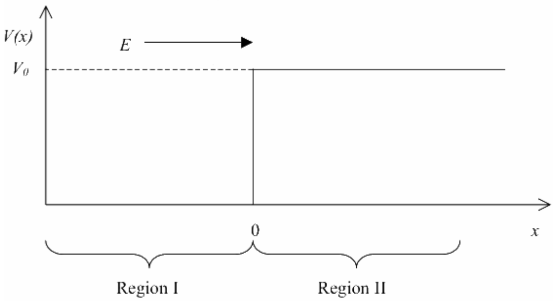
Quantum mechanical potential step

Consider a flux of particles with total energy E incident from the left on a square potential step. Assume $E > V_0$.

- $x < 0$ (Region I):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

$$\Rightarrow \psi''(x) = -k_1^2\psi(x) \quad k_1^2 = \frac{2mE}{\hbar^2} > 0$$



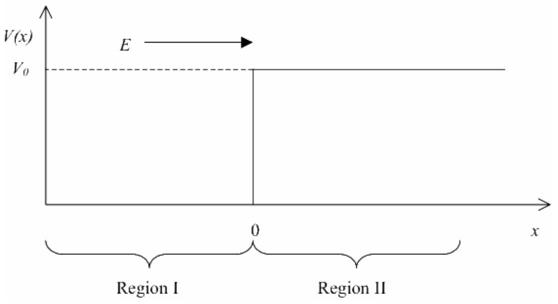
The physical solutions are

KE: $T_1 = \frac{\hbar^2 k_1^2}{2m} = E$

$$\boxed{\psi_I(x) = \underbrace{A\exp(ik_1x)}_{\text{incident}}}$$

Quantum mechanical potential step

Consider a flux of particles with total energy E incident from the left on a square potential step. Assume $E > V_0$.



- $x < 0$ (Region I):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

$$\Rightarrow \quad \psi''(x) = -k_1^2\psi(x) \qquad k_1^2 = \frac{2mE}{\hbar^2} > 0$$

The physical solutions are

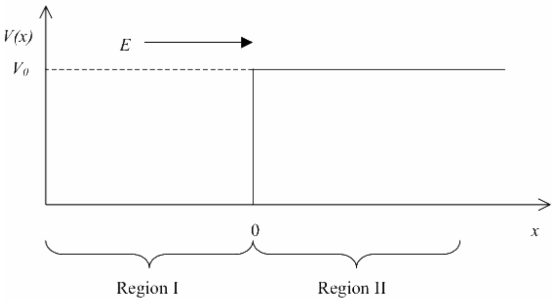
KE: $T_1 = \frac{\hbar^2 k_1^2}{2m} = E$

$$\boxed{\psi_I(x) = \underbrace{A\exp(ik_1x)}_{\text{incident}} + \underbrace{R\exp(-ik_1x)}_{\text{reflected}}}$$

Quantum mechanical potential step

Consider a flux of particles with total energy E incident from the left on a square potential step. Assume $E > V_0$.

- $x < 0$ (Region I):



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

$$\Rightarrow \psi''(x) = -k_1^2\psi(x) \quad k_1^2 = \frac{2mE}{\hbar^2} > 0$$

The physical solutions are

KE: $T_1 = \frac{\hbar^2 k_1^2}{2m} = E$

$$\boxed{\psi_I(x) = \underbrace{A\exp(ik_1x)}_{\text{incident}} + \underbrace{R\exp(-ik_1x)}_{\text{reflected}}}$$

A, R – (complex) amplitudes

- $x \geq 0$ (Region II):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V_0\psi(x) = E\psi(x)$$

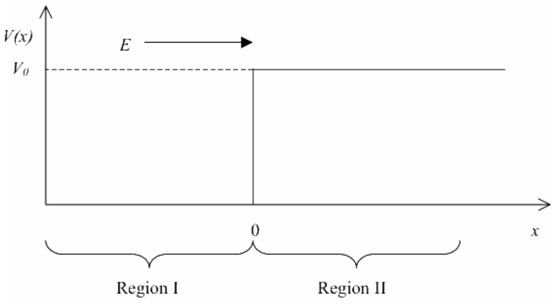
$$\Rightarrow \psi''(x) = -k_2^2\psi(x) \quad k_2^2 = \frac{2m(E-V_0)}{\hbar^2} > 0 \qquad k_2 < k_1$$

KE: $T_2 = \frac{\hbar^2 k_2^2}{2m} = E - V_0 > 0$

Quantum mechanical potential step

Consider a flux of particles with total energy E incident from the left on a square potential step. Assume $E > V_0$.

- $x < 0$ (Region I):



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

$$\Rightarrow \psi''(x) = -k_1^2\psi(x) \quad k_1^2 = \frac{2mE}{\hbar^2} > 0$$

The physical solutions are

KE: $T_1 = \frac{\hbar^2 k_1^2}{2m} = E$

$$\boxed{\psi_I(x) = \underbrace{A\exp(ik_1x)}_{\text{incident}} + \underbrace{R\exp(-ik_1x)}_{\text{reflected}}}$$

A, R – (complex) amplitudes

- $x \geq 0$ (Region II):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V_0\psi(x) = E\psi(x)$$

$$\Rightarrow \psi''(x) = -k_2^2\psi(x) \quad k_2^2 = \frac{2m(E-V_0)}{\hbar^2} > 0 \qquad k_2 < k_1$$

KE: $T_2 = \frac{\hbar^2 k_2^2}{2m} = E - V_0 > 0$

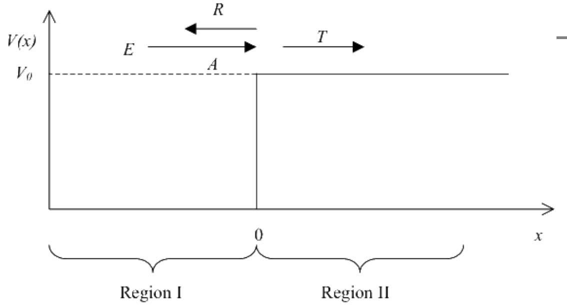
The physical solution is

$$\boxed{\psi_{II}(x) = \underbrace{T\exp(ik_2x)}_{\text{transmitted}}}$$

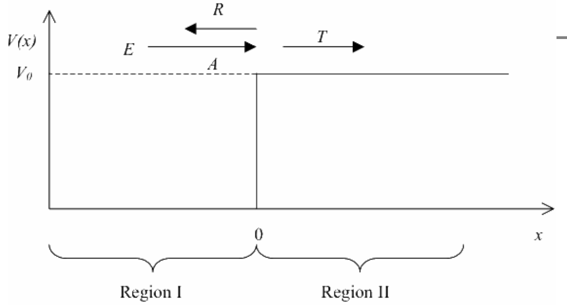
T – (complex) amplitude

Reflection and transmission

Compute the currents:



Reflection and transmission



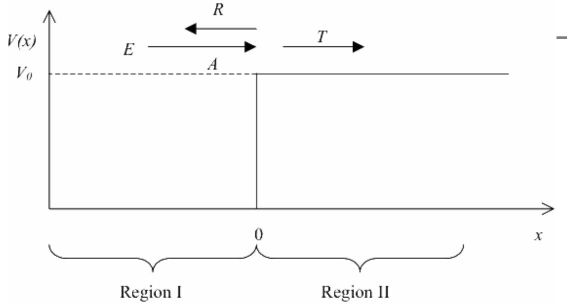
Compute the currents:

- $x < 0$

$$
\begin{aligned}
j_I &= |A|^2 \frac{p_1}{m} &&= |A|^2 \frac{k_1 \hbar}{m} && \textbf{incident beam} \\
j_R &= -|R|^2 \frac{p_1}{m} &&= -|R|^2 \frac{k_1 \hbar}{m} && \textbf{reflected beam}
\end{aligned}
$$

Note that $j_I > 0$ but $j_R < 0$

Reflection and transmission



Compute the currents:

- $x < 0$

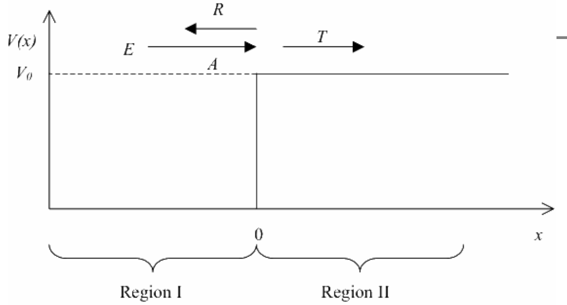
$$
\begin{aligned}
j_I &= |A|^2 \frac{p_1}{m} &= |A|^2 \frac{k_1 \hbar}{m} && \text{incident beam} \\
j_R &= -|R|^2 \frac{p_1}{m} &= -|R|^2 \frac{k_1 \hbar}{m} && \text{reflected beam}
\end{aligned}
$$

- $x \geq 0$

Note that $j_I > 0$ but $j_R < 0$

$$
j_T = |T|^2 \frac{p_2}{m} = |T|^2 \frac{k_2 \hbar}{m} \qquad \text{transmitted beam}
$$

Reflection and transmission



Compute the currents:

- $x < 0$

$$\begin{aligned} j_I &= |A|^2 \frac{p_1}{m} &= |A|^2 \frac{k_1 \hbar}{m} \qquad &\text{incident beam} \\ j_R &= -|R|^2 \frac{p_1}{m} &= -|R|^2 \frac{k_1 \hbar}{m} \qquad &\text{reflected beam} \end{aligned}$$

- $x \geq 0$

Note that $j_I > 0$ but $j_R < 0$

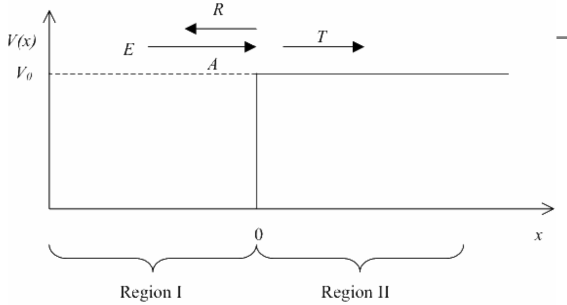
$$j_T = |T|^2 \frac{p_2}{m} = |T|^2 \frac{k_2 \hbar}{m} \qquad \text{transmitted beam}$$

Introduce:

- the probability of reflection P_R

$$P_R = \frac{|j_R|}{|j_I|} = \frac{|R|^2 k_1 \hbar / m}{|A|^2 k_1 \hbar / m} = \frac{|R|^2}{|A|^2} = |\frac{R}{A}|^2$$

Reflection and transmission



Compute the currents:

- $x < 0$

$$j_I = |A|^2 \frac{p_1}{m} = |A|^2 \frac{k_1 \hbar}{m} \qquad \textbf{incident beam}$$

$$j_R = -|R|^2 \frac{p_1}{m} = -|R|^2 \frac{k_1 \hbar}{m} \qquad \textbf{reflected beam}$$

- $x \geq 0$

Note that $j_I > 0$ but $j_R < 0$

$$j_T = |T|^2 \frac{p_2}{m} = |T|^2 \frac{k_2 \hbar}{m} \qquad \textbf{transmitted beam}$$

Introduce:

- the probability of reflection P_R

$$P_R = \frac{|j_R|}{|j_I|} = \frac{|R|^2 k_1 \hbar / m}{|A|^2 k_1 \hbar / m} = \frac{|R|^2}{|A|^2} = |\frac{R}{A}|^2$$

- the probability of transmission P_T

$$P_T = \frac{|j_T|}{|j_I|} = \frac{|T|^2 k_2 \hbar / m}{|A|^2 k_1 \hbar / m} = \frac{|T|^2}{|A|^2} \frac{k_2}{k_1} = |\frac{T}{A}|^2 \frac{k_2}{k_1}$$

Reflection and transmission

Apply boundary conditions on the wavefunction at $x = 0$:

Reflection and transmission

Apply boundary conditions on the wavefunction at $x = 0$:

$$\begin{aligned} \psi_I(x=0) = \psi_{II}(x=0) \quad &\Rightarrow \quad A + R = T \\ &\Rightarrow \quad ik_1(A - R) = ik_2 T \end{aligned}$$

Reflection and transmission

Apply boundary conditions on the wavefunction at $x = 0$:

$$\begin{aligned} \psi_I(x=0) = \psi_{II}(x=0) \quad &\Rightarrow \quad A + R = T \\ \psi'_I(x=0) = \psi'_{II}(x=0) \quad &\Rightarrow \quad ik_1(A - R) = ik_2 T \end{aligned}$$

Reflection and transmission

Apply boundary conditions on the wavefunction at $x = 0$:

$$\begin{aligned} \psi_I(x=0) = \psi_{II}(x=0) \quad &\Rightarrow \quad A + R = T \\ \psi'_I(x=0) = \psi'_{II}(x=0) \quad &\Rightarrow \quad ik_1(A - R) = ik_2 T \end{aligned}$$

Eliminate T: $\frac{R}{A} = \frac{k_1 - k_2}{k_1 + k_2}$

Reflection and transmission

Apply boundary conditions on the wavefunction at $x = 0$:

$$\begin{aligned} \psi_I(x=0) = \psi_{II}(x=0) \quad &\Rightarrow \quad A + R = T \\ \psi'_I(x=0) = \psi'_{II}(x=0) \quad &\Rightarrow \quad ik_1(A - R) = ik_2 T \end{aligned}$$

Eliminate T: $\frac{R}{A} = \frac{k_1 - k_2}{k_1 + k_2}$

Eliminate R: $\frac{T}{A} = \frac{2k_1}{k_1 + k_2}$

Reflection and transmission

Apply boundary conditions on the wavefunction at $x = 0$:

$$\psi_I(x=0) = \psi_{II}(x=0) \quad \Rightarrow \quad A + R = T$$

$$\psi'_I(x=0) = \psi'_{II}(x=0) \quad \Rightarrow \quad ik_1(A - R) = ik_2 T$$

Eliminate T: $\frac{R}{A} = \frac{k_1 - k_2}{k_1 + k_2}$

Eliminate R: $\frac{T}{A} = \frac{2k_1}{k_1 + k_2}$

This leads to:

$$\begin{aligned} P_R &= \left|\frac{R}{A}\right|^2 &&= \frac{(k_1-k_2)^2}{(k_1+k_2)^2} \\ P_T &= \left|\frac{T}{A}\right|^2 \frac{k_2}{k_1} &&= \frac{4k_1k_2}{(k_1+k_2)^2} \end{aligned}$$

Reflection and transmission

Apply boundary conditions on the wavefunction at $x = 0$:

$$\psi_I(x=0) = \psi_{II}(x=0) \quad \Rightarrow \quad A + R = T$$

$$\psi'_I(x=0) = \psi'_{II}(x=0) \quad \Rightarrow \quad ik_1(A - R) = ik_2 T$$

Eliminate T: $\frac{R}{A} = \frac{k_1 - k_2}{k_1 + k_2}$

Eliminate R: $\frac{T}{A} = \frac{2k_1}{k_1 + k_2}$

This leads to:

$$\begin{aligned} P_R &= \left|\frac{R}{A}\right|^2 &= \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \\ P_T &= \left|\frac{T}{A}\right|^2 \frac{k_2}{k_1} &= \frac{4k_1 k_2}{(k_1 + k_2)^2} \end{aligned}$$

Note: $\boxed{P_R + P_T = 1}$

This is a reflection of the fact that the current is conserved,

$$\boxed{j_I + j_R = j_T}$$

(proof)

QM potential step

Comments: $k_1^2 = \frac{2mE}{\hbar^2}$, $k_2^2 = \frac{2m(E-V_0)}{\hbar^2}$

- in the limit of a vanishing step, $V_0 \searrow 0$ or large energy, $E \gg V_0$:

$$k_2 \nearrow k_1 \quad \Rightarrow \quad P_R \searrow 0 \quad P_T \nearrow 1$$

QM potential step - cont.

QM potential step - cont.

Consider next $E < V_0$

QM potential step - cont.

Consider next $E < V_0$

- $x < 0$ (Region I): $k_1^2 = \frac{2mE}{\hbar^2} > 0$

  The solutions are the same as for $E > V_0$,

  $$\boxed{\psi_I(x) = \underbrace{A\exp(ik_1x)}_{\text{incident}} + \underbrace{R\exp(-ik_1x)}_{\text{reflected}}}$$

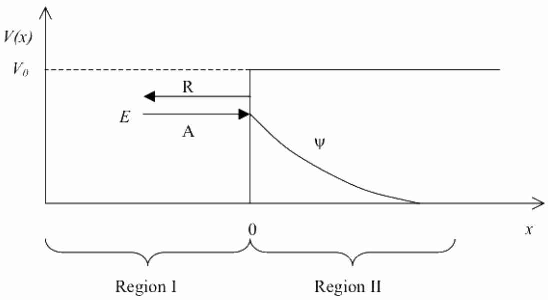
QM potential step - cont.

Consider next $E < V_0$

- $x < 0$ (Region I): $k_1^2 = \frac{2mE}{\hbar^2} > 0$

  The solutions are the same as for $E > V_0$,

$$\boxed{\psi_I(x) = \underbrace{A\exp(ik_1x)}_{\text{incident}} + \underbrace{R\exp(-ik_1x)}_{\text{reflected}}}$$



- $x \geq 0$ (Region II):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V_0\psi(x) = E\psi(x)$$

$$\Rightarrow \psi''(x) = \alpha^2\psi(x) \quad \alpha^2 = \frac{2m(V_0-E)}{\hbar^2} > 0$$

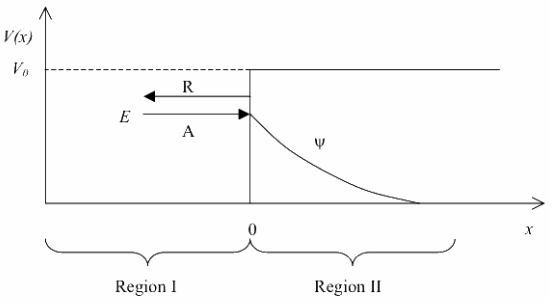
The solution is

$$\boxed{\psi_{II}(x) = \underbrace{T\exp(-\alpha x)}_{\text{tunnelling}}}$$

KE: $T_2 = E - V_0 < 0$!

QM potential step - cont.

Consider next $E < V_0$



- $x < 0$ (Region I): $\quad k_1^2 = \frac{2mE}{\hbar^2} > 0$

  The solutions are the same as for $E > V_0$,

$$\boxed{\psi_I(x) = \underbrace{A\exp(ik_1x)}_{\text{incident}} + \underbrace{R\exp(-ik_1x)}_{\text{reflected}}}$$

- $x \geq 0$ (Region II):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V_0\psi(x) = E\psi(x)$$

$$\Rightarrow \psi''(x) = \alpha^2\psi(x) \quad \alpha^2 = \frac{2m(V_0-E)}{\hbar^2} > 0$$

  The solution is

$$\boxed{\psi_{II}(x) = \underbrace{T\exp(-\alpha x)}_{\text{tunnelling}}}$$

  KE: $T_2 = E - V_0 < 0$!

- the wavef'n $\psi_{II}(x)$ is real $\Rightarrow$ the current $j_T = 0$

  $\Rightarrow$ the probability of transmission $P_T = 0 \Rightarrow P_R = 1 - P_T = 1$

  $\boxed{\text{The whole beam is reflected.}}$

  as expected

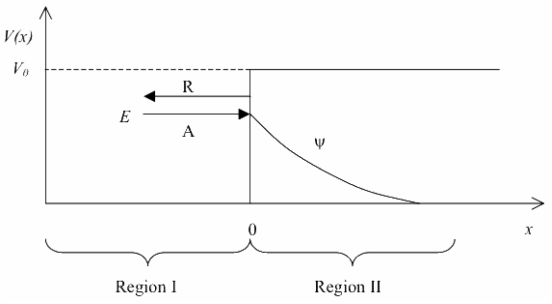
QM potential step - cont.

Consider next $E < V_0$

- $x < 0$ (Region I): $k_1^2 = \frac{2mE}{\hbar^2} > 0$

  The solutions are the same as for $E > V_0$,

$$\boxed{\psi_I(x) = \underbrace{A\exp(ik_1x)}_{\text{incident}} + \underbrace{R\exp(-ik_1x)}_{\text{reflected}}}$$



- $x \geq 0$ (Region II):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V_0\psi(x) = E\psi(x)$$

$$\Rightarrow \ \psi''(x) = \alpha^2\psi(x) \quad \alpha^2 = \frac{2m(V_0-E)}{\hbar^2} > 0$$

The solution is

$$\boxed{\psi_{II}(x) = \underbrace{T\exp(-\alpha x)}_{\text{tunnelling}}}$$

KE: $T_2 = E - V_0 < 0$!

- the wavef'n $\psi_{II}(x)$ is real $\Rightarrow$ the current $j_T = 0$

  $\Rightarrow$ the probability of transmission $P_T = 0 \Rightarrow P_R = 1 - P_T = 1$

  The whole beam is reflected.

  as expected

- the penetration (tunneling) depth is $1/\alpha = \frac{\hbar}{\sqrt{2m(V_0-E)}}$

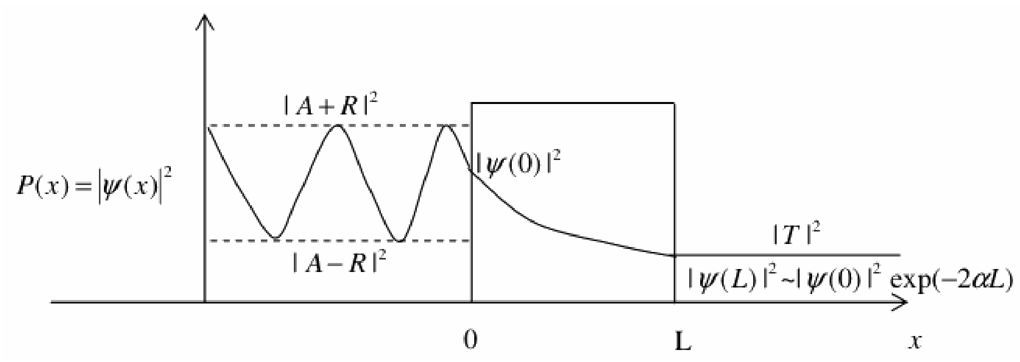
Tunnelling through a barrier

Tunnelling through a barrier

- If the step has a finite length L

Tunnelling through a barrier

- If the step has a finite length L



See also http://yepes.rice.edu/PhysicsApplets/WavePacket.html

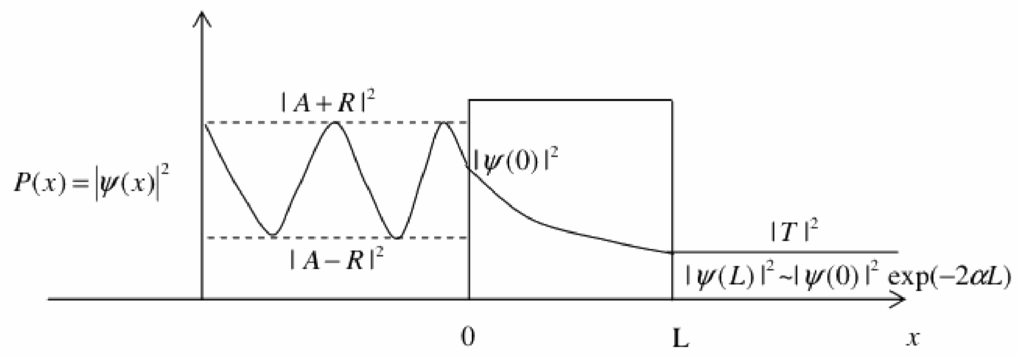
- particle tunnelling between classically allowed regions can take place

$$P_T \simeq |\psi(L)|^2 \simeq |\psi(L)|^2 \exp\left(-2\alpha L\right)$$

$$\alpha^2 = \frac{2m\,(V_0 - E)}{\hbar^2} > 0$$

Tunnelling through a barrier

- If the step has a finite length L



See also http://yepes.rice.edu/PhysicsApplets/WavePacket.html

- particle tunnelling between classically allowed regions can take place

$$P_T \simeq |\psi(L)|^2 \simeq |\psi(L)|^2 \exp(-2\alpha L)$$

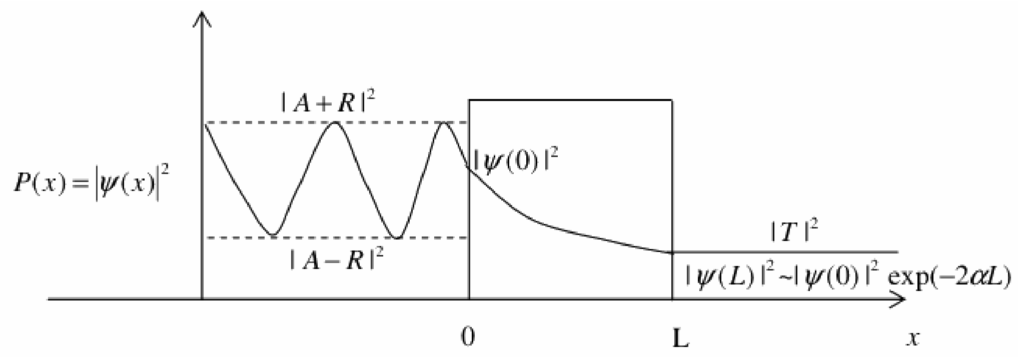
$$\alpha^2 = \frac{2m(V_0 - E)}{\hbar^2} > 0$$

- The HUP for time and energy:
  let ΔE denote the uncertainty in the energy of the particle as it tunnels through the barrier and Δt the time needed to pass through it. Then

$$\boxed{\Delta E \Delta t \geq \frac{\hbar}{2}}$$

Tunnelling through a barrier

- If the step has a finite length L



See also http://yepes.rice.edu/PhysicsApplets/WavePacket.html

- particle tunnelling between classically allowed regions can take place

$$P_T \simeq |\psi(L)|^2 \simeq |\psi(L)|^2 \exp(-2\alpha L)$$

$$\alpha^2 = \frac{2m(V_0 - E)}{\hbar^2} > 0$$

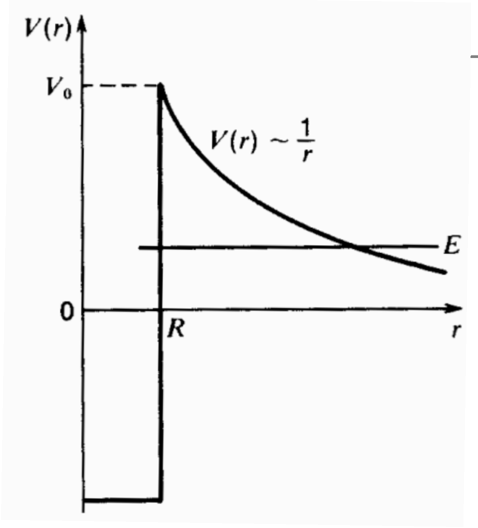
- The HUP for time and energy:
  let ΔE denote the uncertainty in the energy of the particle as it tunnels through the barrier and Δt the time needed to pass through it. Then

$$\boxed{\Delta E \Delta t \geq \frac{\hbar}{2}}$$

- during tunnelling the conservation of energy is violated but the HUP prevents us from observing it experimentally

Tunnelling in nature - examples

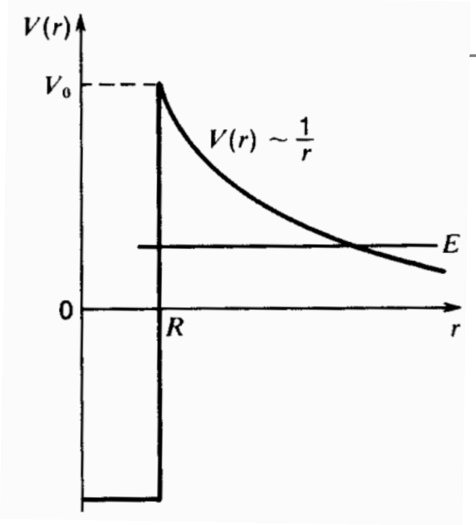
Tunnelling in nature - examples



- α-decay of heavy nuclei

  α particles (^{4}He) tunnel through a nuclear potential barrier arising from the Coulomb repulsion

Tunnelling in nature - examples



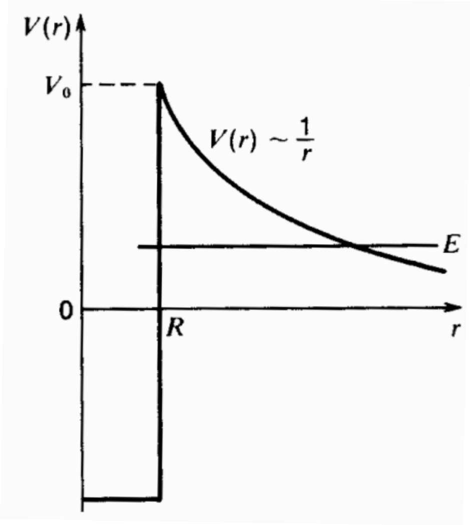
- α-decay of heavy nuclei

  α particles (^{4}He) tunnel through a nuclear potential barrier arising from the Coulomb repulsion

- nuclear fusion of light nuclei

  Nuclear fusion is essentially the inverse of Îś-decay. Here light nuclei tunnel through the Coulomb barrier to form a strongly bound state. Quantum mechanical tunnelling is necessary to explain nuclear fusion in stars.

Tunnelling in nature - examples



- α-decay of heavy nuclei

  α particles (^{4}He) tunnel through a nuclear potential barrier arising from the Coulomb repulsion

- nuclear fusion of light nuclei

  Nuclear fusion is essentially the inverse of Îś-decay. Here light nuclei tunnel through the Coulomb barrier to form a strongly bound state. Quantum mechanical tunnelling is necessary to explain nuclear fusion in stars.

- covalent bonding

  Chemical bonding occurs via the tunnelling of electrons between nuclei. The electrons “resonate” between nuclei, thus lowering their energy and forming a bond.