PHY202 – Quantum Mechanics

Problem class set 2

Eigenfunctions; average values

1. Consider an infinite potential well that is zero in the region $0 \leq x \leq L$ and infinite otherwise. The total energy eigenfunctions for this well are given by $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$, $n = 1, 2, 3, \ldots$, and the corresponding energy eigenstates are given by $E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$.

(a) For any $\psi_n(x)$ evaluate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$ and $\langle p^2 \rangle$, where x stands for position and p for momentum. Where applicable, use the orthonormality properties of the ψ_n's. [4]

(b) Use the results obtained in (a) to evaluate $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$, $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$ and $\Delta x \cdot \Delta p$. Compute its numerical values in units of $\hbar$ for $n = 1, 2$ and 10. Compare them with the Heisenberg Uncertainty Principle for position and momentum and comment on the result. [2]

A particle of mass m inside the above potential well is in a state described by a wavefunction

$$\psi(x) = \frac{2}{\sqrt{L}} \sin\left(\frac{4\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right).$$

(c) Express the wavefunction $\psi(x)$ in terms of the total energy eigenfunctions $\psi_n(x)$ and find the coefficients of expansion a_n. Hence evaluate $\langle x \rangle$ and $\langle p \rangle$ for the state described by the wavefunction $\psi(x)$. [4]

You can use the following integrals:

$\int dx\, x \sin^2 ax = \frac{x^2}{4} - \frac{x}{4a} \sin 2ax + \frac{1}{8a^2} \cos 2ax$,

$\int dx\, x^2 \sin^2 ax = \frac{x^3}{6} - \frac{x^2}{4a} \sin 2ax - \frac{x}{4a^2} \cos 2ax + \frac{1}{8a^3} \sin 2ax$,

$\int dx\, \sin ax \cos ax = -\frac{1}{4a} \cos 2ax$,

$\int dx\, x \sin ax \sin bx = \frac{1}{2(a-b)} \sin 2(a-b)x - \frac{1}{2(a+b)} \sin (a+b)x$, where $a \neq b$

$\int dx\, \sin ax \cos bx = -\frac{1}{2(a-b)} \cos (a-b)x - \frac{1}{2(a+b)} \cos (a+b)x$, where $a \neq b$.

Average value of energy; parity

2. A particle of mass m is confined to a potential that is zero in the region $-L/2 \leq x \leq L/2$ and infinite otherwise. At a certain time its wavefunction is

$$\psi(x) = \sqrt{\frac{2}{5L}} \cos\left(\frac{\pi x}{L}\right) + 2\sqrt{\frac{2}{5L}} \sin\left(\frac{2\pi x}{L}\right).$$

(a) Write down $\psi(x)$ in terms of the total energy eigenfunctions $\psi_n(x)$ found in class. [1]

(b) What are the possible results of the measurement of the energy of this system and what are their relative probabilities? Compute $\langle E \rangle$. [2]

(c) If the energy is immediately remeasured, what will now be the possible outcomes and with what probability? [1]

(d) Does the wavefunction $\psi(x)$ have a definite parity? Explain your answer. [1]

Commutators

3. In the representation such that $\hat{p} = -i\hbar\frac{d}{d\,x}$, $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x) = -\frac{\hbar^2}{2m}\frac{d^2}{d\,x^2} + V(x)$, $\hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{d\,x^2}$, for an arbitrary function $\psi(x)$ evaluate the following commutators:

(a) $[\hat{p}, \hat{H}]\psi(x)$ for any potential $V(x)$; [3]

(b) $[\hat{p}^2, \hat{T}]\psi(x)$, [2]

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